Exercise Sheet #8

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P1. This exercise shows that the locally finite assumption cannot be removed in Theorem 6.15. (The measure μ in part **(b)** is not locally finite). Let $(X,\mathcal{U})$ be the topological space defined by $X := \mathbb{N} \cup \{\infty\}$ and

$$\mathcal{U} := \{U \subset X \mid U \subset \mathbb{N} \text{ or } |U^c| < \infty\}.$$

Thus $(X,\mathcal{U})$ is the (Alexandrov) one-point compactification of the set $\mathbb{N}$ of natural numbers with the discrete topology. (If $\infty \in U$ then the condition $|U^c| < \infty$ is equivalent to the assertion that U^c is compact).

(a) Prove that $(X,\mathcal{U})$ is a compact Hausdorff space and that every subset of X is σ-compact. Prove that the Borel σ-algebra of X is $\mathcal{B} = 2^X$.

> ***Solution:*** First we see that X is compact. Let $X = \bigcup_{i\in I} U_i$ be an open covering. Withoug loss of generality, $\infty \in U_1$. As U_1 is open and not contained in $\mathbb{N}$, we have that $|U_1^c| < \infty$ and thus it is enough to pick finitely many $J \subsetneq I$ such that they $\bigcup_{j\in J} U_j \supset U_1^c$. Thus $U_1 \cup \bigcup_{j\in J} U_j = X$ and X is compact. For checking that X is Hausdorff, for $n, m \in \mathbb{N}$ just take $\{n\}$ and $\{m\}$ as open sets containing n and m.
> Second, let us see that every set of X is σ-compact. Indeed, notice that every finite subset of $\mathbb{N}$ is not only open but also closed. Let $V \subseteq X$. Then, $V = \bigcup_{n\in\mathbb{N}} V \cap \{1, \ldots, n\}$ if $\infty \notin V$ and if $\infty \in V$ then $V = V \setminus \{\infty\} \cup \{\infty\}$ in where $\{\infty\}$ is compact by being complement of the open set $\mathbb{N}$.
> Finally, for proving that the Borel σ-algebra is every possible set. notice that $\mathcal{B}$ already contains $2^{\mathbb{N}}$ as it contains every finite set. Also, it contains $X = \mathbb{N} \cup \{\infty\}$. So, for every $V \subseteq X$ measurable with $\infty \in V$, one can write V as $X \setminus \bigcup_{i\in I} S_i$ where I is countable and S_i are finite subsets of $\mathbb{N}$. Thus, $V \in \mathcal{B}$ and we conclude that $\mathcal{B} = 2^X$.

(b) Let $\mu : 2^X \to [0, \infty]$ be the counting measure. Prove that μ is inner regular, but not outer regular.

> ***Solution:*** Let $E \subseteq X$. If E is infinite, then by approximating by $E \cap \{1, \ldots, e_n\}$ where $e_n \in E$ is the n-th element of E, one gets
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> $$\mu(E) \geq n,$$
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> and thus - making $n \to \infty$ - $\mu(E) = \infty$. If E is finite, then it is compact and the inner regularity follows. For proving that μ is not outer regular, notice that $\mu(\{\infty\}) = 1$ but every open set containing ∞ is infinite and thus of measure ∞, which makes impossible for μ to be inner regular.